

©Journal of Technical University at Plovdiv
 Fundamental Sciences and Applications, Vol. 1, 1995
Series A-Pure and Applied Mathematics
 Bulgaria, ISSN 1310-8271

BMOA estimates and radial growth of B_ϕ functions

Peyo Stoilov, Roumyana Gesheva, Milena Racheva

Abstract

BMO estimates and the radial growth of Bloch functions have been studied by B. Korenblum [3]. The present paper contains some natural generalizations of these results.

1 Introduction

Let D denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and T - the unit circle $\{z : |z| = 1\}$. The space BMOA is the space of functions $f \in H^1$ for which

$$\|f\|_* = \sup \frac{1}{m(I)} \int_I |f - f_I| dm < \infty, \text{ where}$$

$$f_I = \frac{1}{m(I)} \int_I f dm, \quad I \subseteq T.$$

Here m is normalized Lebesgue measure on T .

It is known that for an analytic functions f in D the following conditions are equivalent (see, for example, [1] or [2]):

a) $f \in BMOA$;

$$b) \|f\|_{BMOA}^2 = \sup_{\xi \in D} \iint_D |f'(z)|^2 \frac{(1 - |z|^2)(1 - |\xi|^2)}{|1 - \bar{\xi}z|^2} dm_2 < \infty,$$

⁰**1991 Mathematics Subject Classification:** Primary 30E20, 30D50.

⁰**Key words and phrases:** Bloch space, BMOA estimates, radial growth of functions.

where m_2 denotes normalized Lebesgue measure in D .

Let $\phi(t)$ be a positive and continuous function on $(0, 1)$.

Let B_ϕ denotes the space of all analytic functions f in D satisfying the condition

$$\|f\|_{B_\phi} = \sup_{z \in D} \frac{(1 - |z|^2) |f'(z)|}{\phi(1 - |z|^2)} < \infty.$$

For $\phi(t) = t^\alpha$, $0 < \alpha \leq 1$, $B_\phi = \Lambda_\alpha$ is the usual Lipschitz class.

In the case $\alpha = 0$, Λ_0 is the Bloch space (usually denoted B).

In this paper some results of B. Korenblum for the Bloch space B are generalized for B_ϕ .

Note that the function $\log(1 - z) \in B$, however $\log^2(1 - z) \notin B$.

2 BMOA estimates for B_ϕ functions and applications

Let $\phi(t)$ satisfies the condition

$$\int_x^1 \frac{\phi^2(t)}{t} dt = g(x) < \infty \text{ for all } 0 < x < 1.$$

If $f \in B_\phi$ we write $f_r(z) \stackrel{\text{def}}{=} f(rz)$ for $0 < r < 1$.

Theorem 1. *Let $f \in B_\phi$. Then*

$$\|f_r\|_{BMOA} \leq \|f\|_{B_\phi} \sqrt{g(1 - r^2)}, \quad 0 < r < 1. \quad (1)$$

Proof. Let $\xi \in D$. Then

$$\iint_D |f'_r(z)|^2 \frac{(1 - |z|^2)(1 - |\xi|^2)}{|1 - \bar{\xi}z|^2} dm_2 =$$

$$\begin{aligned}
 &= r^2 \iint_D \frac{|f'(rz)|^2 (1 - |rz|^2)^2}{\phi^2(1 - |rz|^2)} \cdot \frac{\phi^2(1 - |rz|^2)}{(1 - |rz|^2)^2} \cdot \frac{(1 - |z|^2)(1 - |\xi|^2)}{|1 - \bar{\xi}z|^2} dm_2 \leq \\
 &\leq r^2 \|f\|_{B_\phi}^2 \iint_D \frac{\phi^2(1 - |rz|^2)}{1 - |rz|^2} \cdot \frac{1 - |\xi|^2}{|1 - \bar{\xi}z|^2} dm_2 = \\
 &= 2r^2 \|f\|_{B_\phi}^2 \int_0^1 \int_T \frac{\phi^2(1 - r^2\rho^2)(1 - |\xi|^2)}{(1 - r^2\rho^2) |1 - \bar{\xi}\rho\zeta|^2} \rho dm(\zeta) d\rho \leq \\
 &\leq 2r^2 \|f\|_{B_\phi}^2 \int_T \frac{1 - |\bar{\xi}\rho\zeta|^2}{|1 - \bar{\xi}\rho\zeta|^2} dm(\zeta) \int_0^1 \frac{\phi^2(1 - r^2\rho^2)}{1 - r^2\rho^2} \rho d\rho = \\
 &= \|f\|_{B_\phi}^2 \int_{1-r^2}^1 \frac{\phi^2(t)}{t} dt.
 \end{aligned}$$

Here we used the identity

$$\int_T \frac{1 - |z|^2}{|1 - \zeta z|^2} dm(\zeta) = 1.$$

Therefore,

$$\|f_r\|_{BMOA}^2 \leq \|f\|_{B_\phi}^2 g(1 - r^2).$$

Corollary. *If $f \in B$ then*

$$\|f_r\|_{BMOA} \leq \|f\|_{B_\phi} \sqrt{|\log(1 - r^2)|}, \quad 0 < r < 1.$$

B.Korenblum [3] proved an analogous BMO estimate, applying the Garsia norm.

Theorem 2. *There are positive numerical constants γ and M such that for all $f \in B_\phi$, $f(0) = 0$*

$$\int_T \exp \left(\frac{\gamma |f(r\zeta)|}{\|f\|_{B_\phi} \sqrt{g(1-r^2)}} \right) dm(\zeta) \leq M. \quad (2)$$

Proof. The John-Nirenberg theorem [1, 2] says that there are positive constants c and C such that

$$\frac{m(\zeta \in I : |f(\zeta) - f_I| > \lambda)}{m(I)} \leq C \exp \left(\frac{-c\lambda}{\|f\|_{BMOA}} \right)$$

for all $f \in BMOA$, $\lambda > 0$, $I \subseteq T$.

If $f(0) = 0$ then

$$f_T = \frac{1}{2\pi} \int_T f(\zeta) |d\zeta| = 0$$

and

$$E(\lambda) = m(\zeta \in T : |f(r\zeta)| > \lambda) \leq C \exp \left(\frac{-c\lambda}{\|f\|_{BMOA}} \right). \quad (3)$$

Since $E(\lambda)$ is the distilution function of f , then for all $p > 0$ [1]

$$\int_T |f|^p dm = p \int_0^\infty \lambda^{p-1} E(\lambda) d\lambda. \quad (4)$$

If $0 < \gamma < c$, using (4) and (3), we obtain

$$\int_T \exp \left(\frac{\gamma |f(\zeta)|}{\|f\|_{BMOA}} \right) dm(\zeta) = 1 + \sum_{n \geq 1} \frac{1}{n!} \frac{\gamma^n}{(\|f\|_{BMOA})^n} \int_T |f(\zeta)|^n dm(\zeta) =$$

$$\begin{aligned}
 &= 1 + \sum_{n \geq 1} \frac{1}{n!} \frac{\gamma^n}{(\|f\|_{BMOA})^n} n \int_0^\infty \lambda^{n-1} E(\lambda) d\lambda = \\
 &= 1 + \frac{\gamma}{\|f\|_{BMOA}} \sum_{n \geq 1} \frac{1}{n-1!} \frac{\gamma^{n-1}}{(\|f\|_{BMOA})^{n-1}} \int_0^\infty \lambda^{n-1} E(\lambda) d\lambda = \\
 &= 1 + \frac{\gamma}{\|f\|_{BMOA}} \int_0^\infty E(\lambda) \sum_{n \geq 1} \frac{1}{n-1!} \frac{\gamma^{n-1} \lambda^{n-1}}{(\|f\|_{BMOA})^{n-1}} d\lambda = \\
 &= 1 + \frac{\gamma}{\|f\|_{BMOA}} \int_0^\infty E(\lambda) \exp\left(\frac{\gamma \lambda}{\|f\|_{BMOA}}\right) d\lambda \leq \\
 &\leq 1 + \frac{\gamma}{\|f\|_{BMOA}} C \int_0^\infty \exp\left(\frac{-(c-\gamma)\lambda}{\|f\|_{BMOA}}\right) d\lambda = 1 + \frac{\gamma C}{c-\gamma} \stackrel{def}{=} M < \infty.
 \end{aligned}$$

Putting $f = f_r$ and applying (1), we obtain (2).

Theorem 3. There is a constant γ_1 , such that for every $f \in B_\phi$, $f(0) = 0$

$$\lim_{r \rightarrow 1^-} \sup \frac{|f(r\zeta)|}{\log |\log(1-r)| \sqrt{g(1-r^2)}} \leq \gamma_1 \|f\|_{B_\phi} \quad (5)$$

for almost all $\zeta \in T$.

Proof. Theorem 2 implies that $(0 < r < 1)$

$$\int_0^1 \left(\frac{1}{(1-r) \log^2\left(\frac{e}{1-r}\right)} \int_T \exp\left(\frac{\gamma |f(r\zeta)|}{\|f\|_{B_\phi} \sqrt{g(1-r^2)}}\right) dm(\zeta) \right) dr \leq M.$$

Therefore , for almost all ζ

$$\int_0^1 \left(\frac{1}{(1-r) \log^2 \left(\frac{e}{1-r} \right)} \exp \left(\frac{\gamma |f(r\zeta)|}{\|f\|_{B_\phi} \sqrt{g(1-r^2)}} \right) dr \right) < \infty,$$

which implies that

$$\lim_{r \rightarrow 1^-} \int_r^{(r+1)/2} \frac{1}{(1-\rho) \log^2 \left(\frac{e}{1-\rho} \right)} \exp \left(\frac{\gamma |f(\rho\zeta)|}{\|f\|_{B_\phi} \sqrt{g(1-\rho^2)}} \right) d\rho = 0.$$

Putting $\mu(r, \zeta) = \min \{|f(\rho\zeta)| : r \leq \rho \leq (r+1)/2\}$ we get

$$\begin{aligned} & \int_r^{(r+1)/2} \frac{1}{(1-\rho) \log^2 \left(\frac{e}{1-\rho} \right)} \exp \left(\frac{\gamma |f(\rho\zeta)|}{\|f\|_{B_\phi} \sqrt{g(1-\rho^2)}} \right) d\rho \geq \\ & \geq \int_r^{(r+1)/2} \log^{-2} \left(\frac{e}{1-\rho} \right) \exp \left(\frac{\gamma |f(\rho\zeta)|}{\|f\|_{B_\phi} \sqrt{g(1-\rho^2)}} \right) d\rho \geq \\ & \geq \int_r^{(r+1)/2} \log^{-2} \left(\frac{e}{1-\rho} \right) \exp \left(\frac{\gamma |f(\rho\zeta)|}{\|f\|_{B_\phi} \sqrt{g(1-\rho^2)}} \right) d\rho \geq \\ & \geq \log^{-2} \left(\frac{2e}{1-r} \right) \exp \left(\frac{\gamma \mu(r, \zeta)}{\|f\|_{B_\phi} \sqrt{g((1-r)(3+r)/4)}} \right) > 0. \end{aligned}$$

We used the inequalities

$$\log^{-2} \left(\frac{e}{1-\rho} \right) \geq \log^{-2} \left(\frac{e}{1-(r+1)/2} \right) = \log^{-2} \left(\frac{2e}{1-r} \right),$$

$$g(1 - \rho^2) \leq g(1 - (r + 1/2)^2) = g((1 - r)(3 + r)/4).$$

Then

$$\lim_{r \rightarrow 1^-} \log^{-2} \left(\frac{2e}{1 - \rho} \right) \exp \left(\frac{\gamma \mu(r, \zeta)}{\|f\|_{B_\phi} \sqrt{g((1 - r)(3 + r)/4)}} \right) = 0,$$

which implies

$$\lim_{r \rightarrow 1^-} \left(\frac{\gamma \mu(r, \zeta)}{\|f\|_{B_\phi} \sqrt{g((1 - r)(3 + r)/4)}} - 2 \log \log \frac{2e}{1 - r} \right) = -\infty.$$

Since

$$\lim_{r \rightarrow 1^-} \left(\log \log \frac{1}{1 - r} - \log \log \frac{2e}{1 - r} \right) = 0,$$

it can be seen easily that

$$\begin{aligned} \mu(r, \zeta) &< \gamma_1 \|f\|_{B_\phi} \sqrt{g((1 - r)(3 + r)/4)} \log |\log(1 - r)| \leq \\ &\leq \gamma_1 \|f\|_{B_\phi} \cdot \sqrt{g((1 - r^2)/2)} \log |\log(1 - r)|. \end{aligned} \quad (6)$$

for almost all ζ , r sufficiently close to 1 and $\gamma_1 = 2/\gamma$.

In addition, let

$$\mu(r, \zeta) = |f(r_1 \zeta)|, \quad r \leq r_1 \leq (r + 1)/2.$$

Then

$$|f(r\zeta)| - \mu(r, \zeta) \leq \int_r^{r_1} |f'(\rho\zeta)| d\rho \leq \|f\|_{B_\phi} \int_r^{(r+1)/2} \frac{\phi(1 - \rho^2)}{1 - \rho^2} d\rho =$$

$$\begin{aligned}
&= \frac{\|f\|_{B_\phi}}{2r} \int_{(1-r)(3+r)/4}^{1-r^2} \frac{\phi(t)}{t} dt \leq \frac{\|f\|_{B_\phi}}{2r} \int_{(1-r^2)/2}^{1-r^2} \frac{\phi(t)}{t} dt \leq \\
&\leq \frac{\|f\|_{B_\phi}}{2r} \left(\int_{(1-r^2)/2}^{1-r^2} \frac{\phi^2(t)}{t} dt \right)^{1/2} \leq \frac{\|f\|_{B_\phi}}{2r} \left(\int_{(1-r^2)/2}^1 \frac{\phi^2(t)}{t} dt \right)^{1/2} = \\
&= \frac{\|f\|_{B_\phi}}{2r} \sqrt{g((1-r^2)/2))} .
\end{aligned}$$

Applying (6), we obtain

$$|f(r\zeta)| \leq \|f\|_{B_\phi} \sqrt{g((1-r^2)/2))} \left(\frac{1}{2r} + \gamma_1 \log |\log(1-r)| \right)$$

for almost all ζ and r sufficiently close to 1 ,

which proves (5).

Corollary. (Korenblum [3]) *If $f \in B$, $f(0) = 0$ then*

$$\limsup_{r \rightarrow 1^-} \frac{|f(r\zeta)|}{\sqrt{|\log(1-r)|} \log |\log(1-r)|} \leq k \|f\|_B$$

for almost all $\zeta \in T$, where k is an absolute constant.

References

- [1] J. Garnett. *Bounded analytic functions*. Academic Press, New York, 1981.
- [2] P. Koosis. *Introduction to H^p spaces*. Cambridge Univ. Press, Cambridge, 1980.
- [3] B. Korenblum. *BMO estimates and radial growth of Bloch functions*. Bult. Amer. Math. Sos., 12, 1, 1985, 99 -102.

Department of Mathematics
Technical University
25, Tsanko Dijstabanov,
Plovdiv, Bulgaria
e-mail: peyyyo@mail.bg